THE INFLUENCE OF FREQUENCY UPON THE SELF-INDUCTANCE OF COILS

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When currents of low frequency pass through the wires of a coil, the current distributes itself uniformly over the cross sections of the wires. With increasing frequency, this uniform current density no longer prevails, but, as is well known, at least for straight wires, the current density becomes greater at the surface of the wire at the expense of that of the interior.

The corresponding lines of magnetic force become differently distributed, and in consequence the self-inductance suffers a change. A short calculation will show the direction and amount of the change for circuits in which the curvature of the wire may be assumed negligible, and the theory derived for straight wires used.

The theory of this distribution of the current density in straight wires, which has been thoroughly worked out by Lord Rayleigh[1] and by Stefan,[2] is not applicable without modification to the distribution of current density in coils of wire.

The following argument shows that the effect of increasing frequency is to diminish self-inductance.

We shall assume, according to the theory, that with very high frequency the current flows entirely in the surface of the conductor. In computing the mutual inductance between two parallel circuits, we call their distance apart the logarithmic mean distance of the area of one cross section from the other. For circular areas, and circular lines, when completely outside one another, these distances are the same and equal to the distance between their centers. This

[1] Phil. Mag., **21**, p. 381; 1886. [2] Wied. Ann., **41**, p. 400; 1890.

is also true for annular areas between any two concentric circles, that is to say, circular rings. When computing the self-inductance of a circle of circular cross section (solid tore), we use the value of the logarithmic mean distance $r=0.7788$ times the radius of cross section; but in computing the self-inductance of a circular tube (hollow tore), we must use for r the radius of the cross section.

To compute the self-inductance of n turns of wire, knowing the mutual inductances M_{rs} between each pair of turns, and the self-inductance L of each turn, we proceed by the following formula:

$$L_n = nL + 2(n-1)M_{12} + 2(n-2)M_{13} + \ldots\ldots + 2M_{1n} \qquad (1)$$

The deduction of this formula is an extension of the formula of the self-inductance of two equal coils in series:

$$L_2 = L + 2M_{12} + L$$

Hence, it is evident that however the frequency of oscillation may change, the mutual inductance between pairs of wires remains absolutely unchanged, while any change which does occur is due to that of the terms involving self-inductance only.

This difference is therefore n times the difference between the mutual inductance of two circular filaments when placed at a distance apart equal to $0.7788 \times$ radius of the cross section of the wire used, and when placed at a distance apart equal to the radius. For these are the factors used in computing the self-inductance of a turn for low and high frequencies, respectively. Thus, it is easily seen that since in the last case the self-inductance is less than in the first the influence of rapidly oscillating currents is to diminish the coefficient of self-inductance.

Calculations show that this difference is not negligible, so that in accurate work it is necessary to make a correction to the value calculated for low frequencies; this correction being a function of the frequency, the conductivity of the wire, the permeability of the materials of the coil, and of its configuration.

2. Lord Rayleigh has shown[3] that for a straight conductor traversed by sinusoidal currents, the resistance and self-inductance

[3] Loc. cit. See also Gray, Absolute Measurements in Electricity and Magnetism, Vol. 2, part I, p. 325 ff.

for any frequency are given by the following expressions in infinite series:

$$R' = R\left\{1 + \frac{1}{12}B^2 - \frac{1}{180}B^4 + \ldots\ldots\right\}$$

and (2)

$$L' = l\left\{A + \mu\left(\tfrac{1}{2} - \frac{1}{48}B^2 + \frac{13}{8640}B^4 - \ \ldots\right\}\right.$$

where $B = \dfrac{\mu l\omega}{R}$,

R is the resistance for steady currents,
μ is the magnetic permeability of the conductor,
l is the length of conductor,
ω is $2\pi \times$ frequency of the current.

A is determined from the following formula for the self-inductance for currents of low frequency:

$$L = l(A + \tfrac{1}{2}\mu),$$

$$R = \frac{l}{\pi\sigma\rho^2}, \text{ the resistance for steady currents,}$$

where ρ is the radius of the wire and σ is the specific conductivity of the wire.

When the frequency is great, or rather when the product of μ by the frequency is great, these expressions reduce to

$$R' = \sqrt{\tfrac{1}{2}\mu l\omega R}$$

and (3)

$$L' = l\left(A + \sqrt{\frac{\mu R}{2\omega l}}\right)$$

For any frequency we can calculate the apparent resistance of the wire, and also the new value of the self-inductance, thus obtaining an estimate of the variation in self-inductance and resistance with any required frequency.

Calculations were made of this so-called skin effect with the following results:

TABLE I.

Effect of Frequency upon Resistance of Straight Wires.

A Frequency of current	B Thickness of Annulus Copper $\mu=1$	C Thickness of Annulus Iron $\mu=300$
80	0.719	0.0976
120	.587	.0798
160	.509	.0691
200	.455	.0617
480	.293	.0399
800	.228	.0309
1,000	.203	.0276
1,800	.152	.0206
3,200	.114	.0154
4,000	.102	.0138
9,000	.068	.0092
16,000	.051	.0069
20,000	.045	.0062
36,000	.034	.0046
40,000	.032	.0044
64,000	.024	.0035

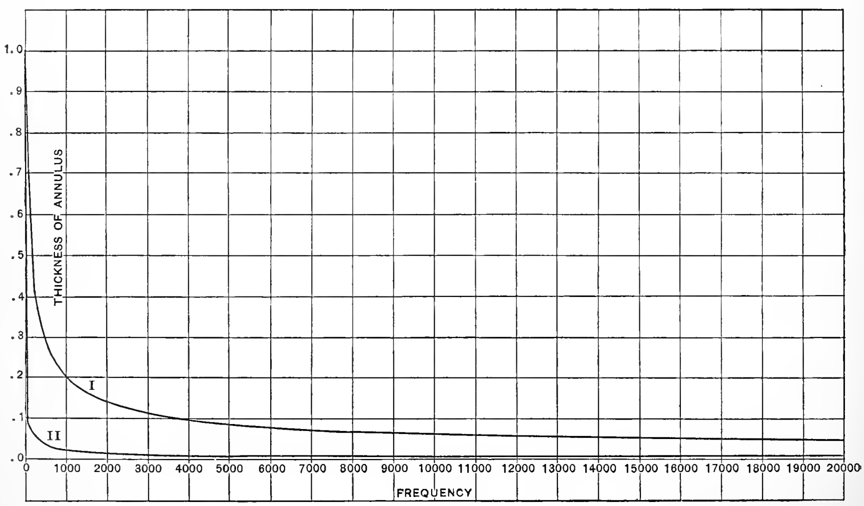

Fig. 1.—*Curves showing the relation between skin-effect and frequency of alternation. Curve I for copper, $\mu=1$; curve II for iron, $\mu=300$.*

The interpretation of Table I is as follows: As the frequency increases so does the resistance. Take a solid wire of any radius, and for a given frequency construct a hollow wire with the same external diameter, which would have the same resistance for steady currents as the apparent resistance of the solid one for alternating currents. Columns B and C give the ratios of the thicknesses of such hollow cylinders to the radius, for copper and iron, respectively.

Fig. 1 gives a graphical representation of these values. The abscissas represent the frequencies. It is seen how readily the current betakes itself to the surface of the conductor, so to speak. It shows that when the material is magnetic, a frequency of 1,000 gives a value of the inductance not appreciably different from frequencies of many times that amount. For copper this corresponding point is not reached before frequencies of 4,000 to 10,000, in the case for which these calculations were made.

3. The value of the self inductance for low frequency currents for a length of wire l is

$$L_0' = l\,(A + \tfrac{1}{2}\,\mu) \tag{4}$$

and the value for very high frequencies is shown by eq. (3) to be

$$L_\infty = lA = 2l\left(\log\frac{2l}{\rho} - 1\right) \tag{5}$$

so that the difference is $\frac{1}{2}\,l\,\mu$; this difference is therefore directly proportional to μ. The coefficient of self-inductance of a straight conductor of circular cross section is given by

$$L = 2l\left\{\log\frac{2l}{\rho} - (1 + \log 0.7788)\right\}$$

where ρ is the radius of the cross section of the conductor; which reduces to

$$L = 2l\left\{\log\frac{2l}{\rho} - \frac{3}{4}\right\} \tag{6}$$

In the theory of the logarithmic mean distance, it is proved that the coefficient of self-inductance of any straight conductor of any cross section is equal to the coefficient of mutual inductance of two parallel straight filaments, at a distance apart equal to the

logarithmic mean distance of the cross section of this conductor from itself. For coils, the assumption is made that when the dimensions of the conductor are small in comparison with the radius of the coil, this law still holds.

Max Wien[4] has shown that this assumption is very close to the truth, by the following example:

By integration of Maxwell's series for the mutual inductance between two parallel circles, over a circular cross section, that is by integration of

$$M=4\pi a \log \frac{8a}{r}\left(1+\frac{x}{2a}+\frac{x^2+3y^2}{16a^2}-\frac{x^3+3xy^2}{32a^3}+\ldots\ldots\right)$$

$$+4\pi a\left(-2-\frac{x}{2a}+\frac{3x^2-y^2}{16a^2}-\frac{x^3-6xy^2}{48a^3}+\ldots\ldots\right) \qquad (7)$$

where the radii of the two circles are a and $a+x$, and where y is the distance between their planes, and r the shortest distance between them, he obtains the following formula for the self-inductance of a circular coil of circular cross section:

$$L=4\pi a\left\{\left(1+\frac{\rho^2}{8a^2}\right)\log\frac{8a}{\rho}-1.75-0.0083\frac{\rho^2}{a^2}\right\} \qquad (8)$$

where ρ is the radius of the cross section of the wires and a is the mean radius of the coil. By employing the same number of terms of the series (7), and putting for y in the series 0.7788 times r he obtains, without integration, the following formula:

$$L=4\pi a\left\{\left(1+\frac{0.91\rho^2}{8a^2}\right)\log\frac{8a}{\rho}-1.75-0.0095\frac{\rho^2}{a^2}\right\} \qquad (9)$$

Assuming, now, such dimensions that ρ, the radius of the circular cross section of the wires is equal to one-fourth the mean radius a, the difference between the two values of L, computed by these formulas, is less than one part in a thousand. This is a severe test, as, in the application of the theory of the logarithmic mean distance, the dimensions of the cross section are assumed to be small in comparison with the mean radius. So that, in the use of this principle,

[4] Ann. d. Phys. und Chem., **53**, p. 928; 1894.

we need not have the slightest fear of inaccuracy if the dimensions satisfy the above condition.

For example, in the application of the principle to the standard coil at Clark University, the difference between the two formulas is given by

$$4\pi a\{.00000437\}$$

which is negligible, producing a difference only in the eighth place of significant figures.

4. Using the first terms of Maxwell's series, i. e., neglecting squares of the ratio $\frac{y}{a}$ we obtain for the mutual inductance of two circles of the same radius at a distance y apart, the expression

$$M=4\pi a\left\{\log\frac{8a}{y}-2\right\}$$

Consider all coils to be wound in the form of a circle, and let the adjective applied refer to the form of the cross section. Then, for a solid circular coil, using the principle of the logarithmic mean distance

$$L=4\pi a\left\{\log\frac{8a}{r}-1.75\right\}$$

and for a circular tube

$$L=4\pi a\left\{\log\frac{8a}{r}-2\right\}$$

Their difference is πa, so that the *self-inductance of a single turn of wire and approximately of any coil of wire, diminishes by an amount numerically equal to half the length of the wire as the frequency becomes infinitely great.* The following computation shows the amount of this decrease for the Clark University coil:

$$\text{Putting } a=27.09 \text{ cm}$$

$$2r= \quad .0584 \text{ cm}$$

the difference πa in the self-inductance of one turn is 85 cm in 2,434 cm or about 3.5 per cent of the value for steady currents.

Now, in any coil the mutual inductances between turns are independent of the distribution of the currents in the separate wires, as

long as this distribution is symmetrical with respect to the axes of the wires; and that is here assumed. The total self-inductance of the coil in question is about 0.2162 times 10^9 cm, and there are 716 turns. Hence, the total change for infinite frequency will be 716 times 85, or 60860 cm, which is about 0.286 per cent of the total value. This certainly is not negligible. It is necessary, then, to look more closely into this source of error.

Lord Rayleigh [5] and J. Stefan [6] have worked out the theory of this effect for straight wires, and have deduced formulæ showing the increase in resistance and the decrease in self-inductance with frequency. Later observers have assumed the effect to be the same in coils as for straight wires. This is not legitimate, as is clearly shown by Max Wien.[7] In his paper he shows that the tendency of the current is to concentrate itself upon the inside surface of the coil, and he derives two formulæ for the increase of resistance and the decrease in self-inductance in circular coils of single and also of many layers, wound with wires of circular cross section.

These formulæ, however, come out as series which converge only for small values of the frequency. The formula for the decrease in self-inductance has but one term derived, and the calculation of the others seems to be very laborious. It is evident from this paper, then, that the maximum change in self-inductance, for a coil of a single layer, is equal to that which would be produced by decreasing the radius of the coil by an amount equal to the radius of the wire used in winding the coil.

Taking the numerical values for the mean radius and radius of the wire used above, and multiplying the coefficient $\frac{\partial L}{\partial a}=0.00491$ by $da=0.0602$ cm, we obtain 0.0003 henry as the total possible change in an inductance of 0.088 henry. This is about 0.34 per

[5] Loc. cit.

[6] Ann. d. Phys., **14**, p. 1; 1904.

[7] The coefficient $\frac{\partial L}{\partial a}$ was derived by direct calculation for the standard coil constructed for Clark University, by the writer.

The self-inductance was computed for a mean radius a, and then for a mean radius $a+da$; the difference gave of course the value of dL for a radius a; dividing by da, the rate of change of L with respect to a was obtained. See Bulletin of the Bureau of Standards, Vol. 2, No. 1.

cent of the total amount and is certainly not negligible. This number compares very favorably with 0.29 per cent derived by an entirely different process.

The formulæ derived by Wien for a coil of a single layer are as follows:

$$R' = R\{1 + 0.272\,\omega^2\phi^2 - \ldots.\}$$

and

$$L' = R\left\{\phi\left(\frac{r}{\rho} - \frac{256}{45\pi^2}\right) - \omega^2\phi^3\Gamma + \ldots.\right\} \qquad (10)$$

where

R' is the resistance at frequency $\frac{\omega}{2\pi}$,

R is the resistance at frequency 0,

L' is the self-inductance at frequency $\frac{\omega}{2\pi}$,

c is the length of coil,
ρ is the radius of wire,
σ is the specific resistance,
m is the total number of terms,
r is the radius of coil,
Γ is a small constant,
ω is $2\pi\times$ frequency,
ϕ equals $\frac{2\pi^2\rho^3 m}{\sigma c}$.

The first important result of this calculation is that the increase of resistance is thirty-two times as large for coiled wire as it is for the same length of straight wire. This shows the danger of using without modification, for coils of wire, the results of Lord Rayleigh for straight wires.

We should expect a corresponding difference for the change in self-inductance, but from his results this is not possible to verify, as he was principally interested in the change of resistance. Wien's calculations agree very well with the experiments of Dolezalek [8] on the change of resistance for small frequencies.

The next successful attempt to find the change of resistance with frequency was made by A. Sommerfeld,[9] who derived an expression

[8] F. Dolezalek: Ann. d. Phys., **12**, p. 1142; 1903.
[9] A. Sommerfeld: Ann. d. Phys., **15**, p. 673; 1904.

valid for all frequencies and which for small frequencies reduced to that of Max Wien. He also was principally interested in the change of resistance, so that the application of his method to the variation in self-inductance was left untried.

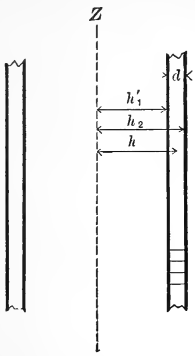

Fig. 2.

In the following will be found the derivation of a formula by a method based on that of Sommerfeld, valid for all frequencies, showing the change in self-inductance with the frequency. By means of this we shall be able to calculate the correction to be applied to any coil of a single layer.

5. *Derivation of the Formula.*—Consider an infinite circular current sheet of thickness d, inside radius r_1, and outside radius r_2.

Let the symbols **S** ( ) and **V** ( ) denote the scalar and vector products respectively. ∇ is the Hamiltonian operator. **V** ∇ ( ) is the same as curl ( ) or rot. ( ), while **S** ∇ ( ) is the same as div. ( ).

By symmetry, the field inside the coil is axial, and a function of r alone, at any given time, and on account of the coil being infinite in length, the field is all inside of it.

In the dielectric, Maxwell's equations are, if we neglect the displacement current,

$$\mathbf{V}\nabla H=o \text{ and } \mathbf{S}\nabla H=o. \qquad (11)$$

From the second relation we find that H is independent of z, and from the first relation, that it is independent of x and y; hence it is constant inside the coil, and for

$$r \overline{\overline{<}} r_1 \text{ assume } H=H_0 e^{i\omega t} \qquad (12)$$

where ω is $2\pi\times$ frequency, and H_0 the maximum value of H. Outside the coil, H must be constant, but as it vanishes at infinity it is zero everywhere. Hence, for

$$r \overline{\overline{>}} r_2,\ H=O.$$

In the material of the coil, Maxwell's equations are

$$4\pi q = \mathbf{V}\nabla H, \ \frac{dH}{dt} = -\mathbf{V}\nabla F \text{ and } q = \sigma F \tag{13}$$

from which follow

$$\mathbf{S}\nabla H = o, \text{ and } \mathbf{S}\nabla q = o. \tag{14}$$

where

q is the current density
F is the electric force
σ is the specific conductivity.

Eliminating q and F, by taking the curl of the first equation (13)

$$4\pi\mathbf{V}\nabla q = \mathbf{V}\nabla\mathbf{V}\nabla H = \nabla\mathbf{S}\nabla H - \nabla^2 H$$

$$= -\nabla^2 H, \text{ by (14)},$$

and from the second and third of (13)

$$\mathbf{V}\nabla q = \sigma\mathbf{V}\nabla F = -\sigma\frac{dH}{dt}$$

$$\therefore\ 4\pi\sigma\frac{dH}{dt} = \nabla^2 H \tag{15}$$

Now assume H in the material of the coil to be given by a function of r alone (on account of symmetry), multiplied by $H_0 e^{i\omega t}$, or $H = H_0 U e^{i\omega t}$

where $U = 1$ when $r = r_1$
$U = 0$ " $r = r_2$

The differential equation then which U must satisfy becomes by equation (15) in polar coordinates

$$\frac{d^2 U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + k^2 U = 0 \tag{16}$$

where $k = -i\,4\pi\sigma\omega$.

The solutions of this equation are the Bessel's functions of order zero, J_0 and K_0, and

$$H = H_0\{AJ_0(kr) + BK_0(kr)\}e^{i\omega t} \tag{17}$$

where A and B are arbitrary constants.

Now, Sommerfeld[10] has shown that the expression

$$H=\sqrt{\frac{r_1}{r}}\frac{e^{ik(r_2-r)}-e^{-ik(r_2-r)}}{e^{ik(r_2-r_1)}-e^{-ik(r_2-r_1)}}e^{i\omega t} \tag{18}$$

is a very good approximation formula for the case in which kr_1 and kr_2 are large.

Since

$$4\pi q=\mathbf{V}\nabla H=-\frac{dH}{dr} \tag{19}$$

we obtain for the total current I per unit of length the value

$$I=\frac{I}{N}\int_{r_1}^{r_2}qdr=-\frac{1}{4\pi N}\int_{r_1}^{r_2}\frac{dH}{dr}dr=\frac{1}{4\pi N}\left(H_{r_1}-H_{r_2}\right)$$

$$\therefore I=\frac{H_0}{4\pi N}e^{i\omega t} \tag{20}$$

where N is the number of turns per unit length of the coil.

6. The electrokinetic energy of a system is given by

$$T=\frac{1}{2}LI^2 \tag{21}$$

where L is its self-inductance, and I is the current. In the case of alternating currents, the mean square values are meant, or

$$T_1=\frac{1}{2}LI_1^2$$

Hence we may write

$$L=\frac{2T_1}{I_1}=2\frac{\frac{1}{\tau}\int_0^\tau Tdt}{\frac{1}{\tau}\int_0^\tau I^2dt} \tag{22}$$

where τ is the period.

But the mean square value of an harmonic function is one-half the square of its amplitude, and in working with imaginary expressions the square of the amplitude can always be obtained by multiplying any expression by its conjugate.

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[10] Loc. cit., p. 678.

Hence

$$I_1^2 = \frac{1}{\tau}\int_0^\tau I^2 dt = \tfrac{1}{2}\, I\,\bar{I} \tag{23}$$

where $\bar{I}$ denotes the conjugate to I.

Therefore

$$\bar{I}^2 = \tfrac{1}{2}\, I\, I = \tfrac{1}{2}\left(\frac{H_0}{4\pi N}\right)^2$$

and

$$L = 4\left(\frac{4\pi N}{H_0}\right)^2 \frac{1}{\tau}\int_0^\tau T dt. \tag{24}$$

In general $$T = \frac{1}{8\pi}\int\int\int_\infty H^2 dv \text{ if } \mu = 1 \text{ everywhere}, \tag{25}$$

which in this case becomes, for unit length of coil,

$$T = \frac{1}{8\pi}\int_0^\infty H^2 2\pi r dr$$

or, $$T = \tfrac{1}{4}\int_0^{r_1} H^2 r dr + \tfrac{1}{4}\int_{r_1}^{r_2} H^2 r dr = T_1 + T_2 \tag{26}$$

where, in T_1, H is $H_0\, e^{i\omega t}$

and in T_2, H is given by equation (18).

We may write

$$\frac{1}{\tau}\int_0^\tau T dt = \tfrac{1}{4}\int r dr\, \frac{1}{\tau}\int_0^\tau H^2 dt = \tfrac{1}{8}\int H\bar{H} r dr.$$

For T_1 this becomes $$\left(\frac{H_0 r_1}{4}\right)^2. \tag{27}$$

To obtain the integral T_2 first put in eq. (18) $h\,(1+i)$ for ik.

Then, calling $h\,(r_2 - r) = y$,

and $$h\,(r_2 - r_1) = y_1$$

we obtain for H and its conjugate $\bar{H}$ the expressions:

$$H = \sqrt{\frac{r_1}{r}}\, \frac{e^{iy}e^{y} - e^{-iy}e^{-y}}{e^{iy_1}e^{y_1} - e^{-iy_1}e^{-y_1}}\, H_0 e^{i\omega t}$$

and (28)

$$\bar{H} = \sqrt{\frac{r_1}{r}}\, \frac{e^{-iy}e^{y} - e^{iy}e^{-y}}{e^{-iy_1}e^{y_1} - e^{iy_1}e^{-y_1}}\, H_0 e^{-i\omega t}$$

A conjugate expression is obtained by putting $-i$ for i everywhere in the equation.

Their product becomes

$$H\overline{H}=H_0^2\frac{r_1}{r}\frac{(e^{2y}+e^{-2y})-(e^{2iy}+e^{-2iy})}{(e^{2y_1}+e^{-2y_1})-(e^{2iy_1}+e^{-2iy_1})}$$

or

$$H\overline{H}=H_0^2\frac{r_1}{r}\frac{\cosh 2y-\cos 2y}{\cosh 2y_1-\cos 2y_1}.$$

Multiplying this expression by rdr and integrating from r_1 to r_2 gives

$$\left(\text{since}\int\cosh r dr=\sinh r\right)$$

putting $$x=2h(r_2-r_1)=2hd,$$

$$T_2=\frac{H_0^2 r_1 d}{8}\frac{\sinh x-\sin x}{x\cosh x-\cos x}.$$

We may now write

$$L_\omega=4\pi^2N^2r_1^2\left\{1+\frac{2}{3}\frac{d}{r_1}\left(\frac{3\sinh x-\sin x}{x\cosh x-\cos x}\right)\right\} \tag{29}$$

where $$x=2d\sqrt{2\pi\omega\sigma}=(4\pi d\sqrt{\sigma})\sqrt{n} \text{ where } \omega=2\pi n.$$

Now, $4\pi^2N^2r_1^2$ is the self-inductance per unit length of a coil with a mean radius r_1, and is therefore the self-inductance for infinite frequency. This is also shown by eq. (29) as the second term in parenthesis is zero when $x=\infty$ for $\frac{\sinh x}{\cosh x}=1$ and $\sinh x$ and $\cosh x$ increase without limit as x increases without limit, as may be seen by their expansions:

$$\begin{aligned}\sin hx&=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots;\quad \sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots;\\ \cos hx&=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots;\quad \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots.\end{aligned} \tag{30}$$

Equation (29) may then be written

$$\frac{L_\omega-L_\infty}{L_\infty}=\frac{2}{3}\frac{d}{r_1}\left(\frac{3\sinh x-\sin x}{x\cosh x-\cos x}\right)=\frac{2}{3}\frac{d}{r_1}Q. \tag{31}$$

By this equation we see that the fractional change of self-inductance in any coil may be calculated if the properties of the function

$$Q = \frac{3}{x}\frac{\sinh x - \sin x}{\cosh x - \cos x} \tag{32}$$

be known for all values of x from zero to infinity.

A formula for small frequencies may be derived by expanding the hyperbolic and circular functions and retaining only low powers of the variable x. Q then becomes

$$Q = 3\frac{\frac{1}{3!}+\frac{x^4}{7!}+\frac{x^8}{11!}+\ \cdots\cdots}{\frac{1}{2!}+\frac{x^4}{6!}+\frac{x^8}{10!}+\ \cdots\cdots}$$

which, retaining only terms in x^4 is

$$Q = \frac{\frac{1}{2}+\frac{3x^4}{7!}}{\frac{1}{2}+\frac{x^4}{6!}} = \frac{1+\frac{6x^4}{7x^1}}{1+\frac{2x^4}{6!}} = 1+\frac{6x^4}{7!}-\frac{14x^4}{7!} = 1-\frac{8x^4}{7!} = 1-\frac{x^4}{630.}$$

This equation is applicable for small frequencies only, and (31) may be written, since $64\pi^2 = 630$ very approximately,

$$\frac{L_\omega - L_\infty}{L_\infty} = \frac{2}{3}\frac{d}{r_1}\left(1 - \omega^2 d^4 \sigma^2\right). \tag{33}$$

It agrees as to form and dimensions with that of Max Wien, eq. (10).

When $\omega = o$, it reduces to $\frac{2}{3}\frac{d}{r_1}.$ As a verification of the term $\frac{2}{3}\frac{d}{r_1}$, the approximation formula for Bessel's formula with small argument

$$H = H_0\sqrt{\frac{r_1}{r}}\frac{r_2 - r}{r_2 - r_1}e^{i\omega t} \tag{34}$$

was taken and a calculation similar to the one above was made for the general case. This completely verified this term, and proved that the expression used above, eq. (31), is valid for small arguments

as well as large. This last expression $\frac{2}{3}\frac{d}{r_1}$ multiplied by $L_\infty = 4\pi^2N^2r_1^2$ is

$$L_0 - L_\infty = \frac{8}{3}\pi^2N^2r_1d \tag{35}$$

and is therefore the additional self-inductance due to the field in the wires themselves for steady values of the current, or, in other words, this is the maximum possible change in self-inductance. Calculations by means of (35) show this expression to give, for coils of finite length and wires with round instead of square cross sections, the following values:

TABLE II.

Comparison of the Theoretical with the True Change in Self-Inductance.

Coil Length	True Change	Value of $\frac{2}{3}\frac{d}{r_1}$	Ratio
46.0 cm.	.00038 henry	.00028 henry	1.35
30.5 "	.00020 "	.00016 "	1.25
15.0 "	.000067 "	.000057 "	1.17
13.0 "	.000051 "	.000044 "	1.16

The values in the second column were computed by taking the difference between the self-inductance of a coil of mean radius a, computed by means of an exact formula, and that of the same coil, using as a mean radius $a-\rho$, where ρ is the radius of the wire. These results, considering the assumptions made in the theory, are in good agreement with it. To make the theory fit the facts, the second term in the parenthesis in equation (29) should be multiplied by a constant, the average value of which is approximately 1.25, deduced from the above table. We should expect some multiplier to be necessary because a current sheet is the equivalent of square wires without insulation, while, in any actual coil, the wires are round and have insulation. We may then write

$$\frac{L_\omega - L_\infty}{L_\infty} = 1.25\,\frac{2}{3}\frac{d}{r_1}\left(\frac{3}{x}\,\frac{\sinh x - \sin x}{\cosh x - \cos x}\right). \tag{36}$$

In the proposed use of this theory, however, it is not necessary to employ this constant, whose value at best is not satisfactorily determinate.

As cosh x and sinh x increase without limit as x increases, equation (36) may be written, for large values of ω

$$\frac{L_\omega - L_\infty}{L_\infty} = 1.25\,\frac{d}{r_1}\,\frac{3}{x} = \frac{5}{4}\,\frac{d}{r_1}\,\frac{1}{d\sqrt{2\pi\omega\sigma}} \tag{37}$$

This formula agrees to within a constant factor with that of Lord Rayleigh for large values of n, equation (3),

or
$$\frac{L_\omega - L_\infty}{L_\infty} = \sqrt{\frac{R}{2nl}} = 2\,\frac{1}{d\sqrt{2\pi\omega\sigma}}.$$

Computations were carefully made for the values of the term

$$Q = \frac{3}{x}\,\frac{\sinh x - \sin x}{\cosh x - \cos x} \tag{38}$$

for the different values of x, as well as of Q_1 and Q_2 where

$$Q_1 = 1 - \frac{x^4}{630}, \tag{39}$$

$$Q_2 = \frac{3}{x}. \tag{40}$$

Table III contains the results.

It is clearly seen that as long as $x<2$ formula Q_1 may be used, while if $x>7$, formula Q_2 is valid. Between these points the calculated values will serve.

A plot of these three curves shows still more clearly the relations of the three formulæ for different values of x. (See Fig. 3.)

7. *Manner of Using the Curve.*—This curve is to be used in the following manner. We assume that in any coil of a single layer the falling off of the self-inductance takes place according to equation (32), represented by the full line curve, and the maximum ordinate unity is taken to be the maximum possible change of the self-inductance, which may be calculated. For any given dimensions of the coil the

TABLE III.

Values for the Curves Q, Q_1, and Q_2 for Values of x from 0 to ∞.

x	cosh x	sinh x	cos x	sin x	Q	Q_1	Q_2
0.0	1.0000	0.0000	1.0000	0.0000	1.0000	1.0000	∞
0.5	1.1276	0.5211	0.8776	0.4789	0.9999	0.9999	6.0000
1.0	1.5431	1.1752	0.5402	0.8415	0.9984	0.9984	3.0000
1.5	2.3523	2.1291	0.0706	0.9975	0.9920	0.9918	2.0000
2.0	3.7622	3.6269	−0.4163	0.9092	0.9756	0.973	1.5000
3.0	10.0677	10.0178	−0.9900	0.1409	0.8932	0.871	1.0000
4.0	27.308	27.262	−0.6534	−0.7570	0.7515	0.594	0.7500
5.0	74.204	74.197	+0.2840	−0.9588	0.6100	Negative	0.6000
6.0	202.approx.	202.approx.	0.9603	−0.2790	0.5032	"	0.5000
7.0	573.approx.	573.approx.	0.7537	0.6574	0.4287	"	0.4289
8.0	Very large		....	....	0.3750	"	0.3750
9.0	and the		....	....	0.3333	"	0.3333
10.0	same as		....	....	0.3000	"	0.3000
	sinh x						
∞	∞	∞	Indeter.	Indeter.	0	− ∞	0

values of x correspond to definite frequencies, and either a table or a plot of corrections may be derived from the curve.

For small values of x, and therefore for small values of the frequency, the correction varies proportionately to d^4, while for very large frequencies it is proportionate to $\frac{1}{d}$ by equations (39) and (40).

As the first case is the more usual, it is important, in order to avoid the necessity for any correction, to wind the coil with wires of small diameter. The size necessary to avoid a correction under a given frequency may easily be calculated. As an example of the employment of this theory, the following example will be carried out:

Consider a coil of 27 cm radius $= r_1$ wound with a single layer of wire, whose diameter d is 0.063 cm. Let the conductivity of the wire be taken as 5.9×10^{-4}. Then the relation between x and n is, from (29) where $n =$ frequency,

$$n = 2630x^2.$$

In the following table, column 2, are the values of x corresponding to the frequencies n in column 1. In column 3 are the values of the ordinates corresponding to these values of x taken from the curve, Fig. 3.

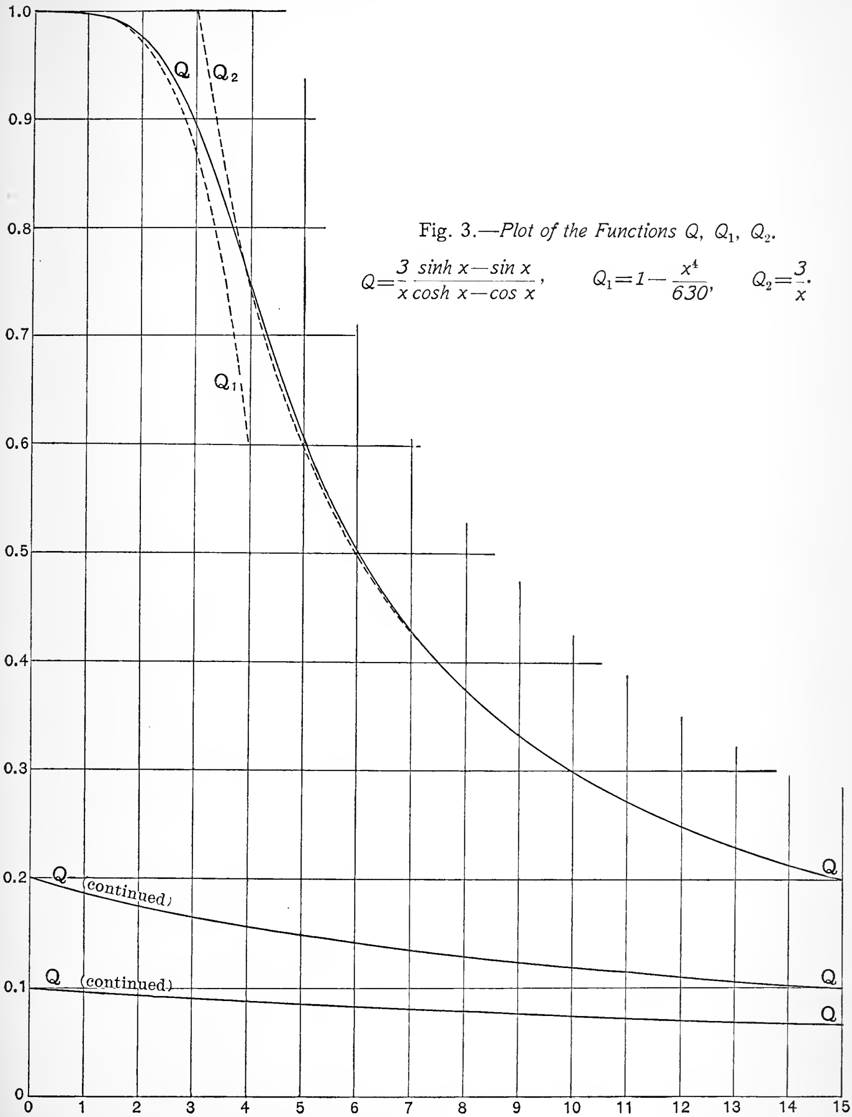

Fig. 3.—*Plot of the Functions* Q, Q_1, Q_2.

$$Q=\frac{3}{x}\,\frac{\sinh x-\sin x}{\cosh x-\cos x},\qquad Q_1=1-\frac{x^4}{630},\qquad Q_2=\frac{3}{x}.$$

TABLE IV.

Values of Frequency n, and Corresponding Values of x and of Relative Change in Self-Inductance.

n	x	Ordinate
0	0	1.000
500	.44	1.000
1,000	.62	0.999
1,500	.76	0.998
3,000	1.07	0.996
4,000	1.23	0.995
5,000	1.38	0.993
6,000	1.51	0.990
10,000	1.95	0.977
15,000	2.38	0.950
20,000	2.75	0.916
30,000	3.38	0.846
40,000	3.90	0.765
50,000	4.35	0.700
60,000	4.78	0.638
70,000	5.15	0.590
80,000	5.51	0.550
90,000	5.85	0.515
100,000	6.17	0.483
120,000	6.76	0.463
150,000	7.55	0.397
170,000	8.05	0.373
200,000	8.72	0.344

The plot of the corresponding values of n as abscissas and values of column 3 as ordinates gives a curve of corrections for the above coil. For example, at a frequency of 50,000 the correction is 0.3 of the maximum change. The value of the correction may be obtained by reading *down* from the ordinate unity.

These computations also show that equation (39)

$$Q_1 = 1 - \frac{x^4}{630}$$

may be employed up to frequencies of 10,000 for this particular coil, and that equation (40)

$$Q_2 = \frac{3}{x}$$

may be used for frequencies beyond 120,000 per second. As the maximum correction is only 0.2 per cent of the total, if frequencies under 4,000 per second are used, the error will not be over 0.2 per cent $\times 0.005 = 0.001$ per cent, which is practically negligible.

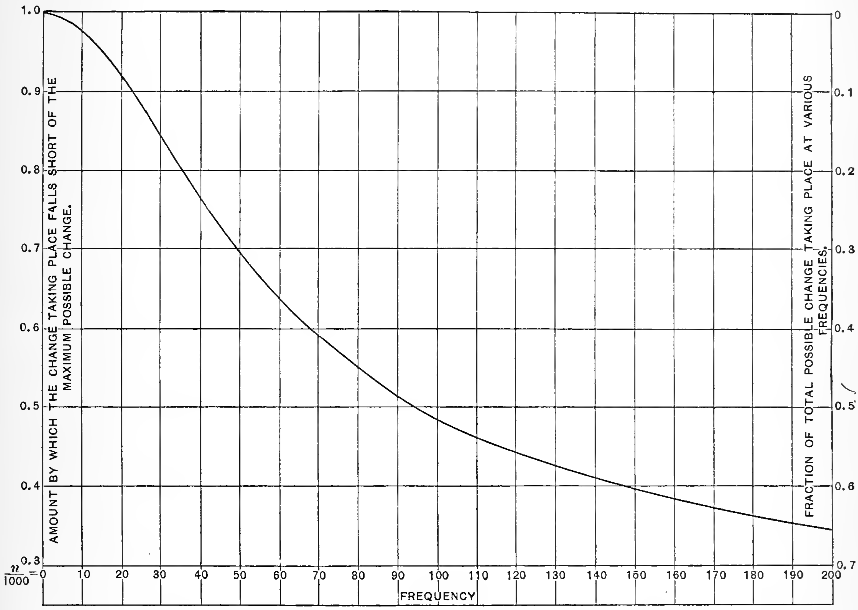

Fig. 4.—*Curve showing the fractional part of the maximum possible decrease in inductance which takes place for different frequencies in a coil the wire of which has a diameter of 0.063 cm, the radius of the coil being 27 cm.*

If, however, other things being equal, the diameter of the wire were four times larger than it is, the same corrections would correspond to frequencies 16 times less, or an error of 0.001 per cent is now obtained at a frequency of only 250.

This suggests that if large currents must be carried, to wind the coil with *flat* wires would diminish the effect of the frequency upon the changes both in self-inductance and in resistance.

Figs. 3 and 4 show how the self-inductance of a circular current sheet falls off with the frequency. In other words, the coil is assumed to be wound with wires of rectangular cross section. This may cause a doubt as to the applicability of the curve in question to the winding of round wires.

In the first place, the method of its use does away with the necessity of knowing the constant multiplier of the function Q, as explained above; and in the second place, the following two reasons go to show the entire correctness of the deductions.

Sommerfeld finds that his results agree to within a constant with those of Max Wien, which were deduced for circular wires, and both formulæ agree remarkably well with the experimental data at hand; this for the increase of resistance with the frequency. We should therefore fairly expect the results for the decrease in self-inductance also to be correct to within a constant factor. As there are no data available for the decrease in self-inductance, we have not been able to verify this conclusion. To further assure ourselves, however, we have worked out the problem of the change of self-inductance with the frequency in an infinite straight circular ring conductor, using the approximation formulæ for the Bessel's functions, as above given. It was found that the variable part of the expression for self-inductance agreed identically with that derived for a circular current sheet. This shows that the bending of a straight wire into a circular shape and forming a coil therewith does not affect the manner in which the self-inductance changes.

We believe, then, there is no doubt that Fig. 3 gives the law according to which the self-inductance falls off for straight ring-shaped conductors and probably for any simply shaped square or polygonal straight conductors, as well as for circular coils of a single layer, wound with wires of circular, square, or polygonal cross sections.